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Solution of Steinhaus's Problem with Plus and Minus Signs

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In a row n plus and minus signs are given. If under each pair of equal signs a positive sign and under opposite signs a minus sign is written, they finally determine a triangle like Figure 1. For all possible n ($n \equiv 0, 3 \pmod 4$) the question whether there exists a first row in such a way that half of all signs of the triangle is positive, is answered in the affirmative.

In [1], H. Steinhaus posed the following problem: There is a triangle of plus and minus signs (see Figure 1). Under each pair of equal or opposite signs a plus or minus sign appears, respectively. So the triangle with $N = n(n+1)/2$ signs is determined by the n signs of the first row. As

```
+ − + + − + − − + + +
 − − + − − − + − + +
  + − − + + − − − +
   − + − + − + + −
    − − − − − + −
     + + + + − −
      + + + − +
       + + − −
        + − +
         − −
          +
```

FIGURE 1

N is even only for $n \equiv 0$ or $3 \pmod 4$, for all these n it is asked for a first row such that as much plus as minus signs occur in the triangle. Figure 1 gives a solution for $n = 11$, from which we obtain solutions for $n = 8, 7, 4,$ and 3 by rejecting the first 3, 4, 7, and 8 rows, respectively. By the way, 11 is the greatest n having a triangle with the same numbers of plus and minus signs, so that all lowest m rows with $m \equiv 0, 3 \pmod 4$ form triangles having also this property.

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We now replace the signs plus and minus by the digits 0 and 1, respectively. If $a_{i,j}$, $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant n - i + 1$, denotes the j-th digit in the i-th row of a triangle, then $a_{i,j}$ is defined by

$$a_{i,j} \equiv a_{i-1,j} + a_{i-1,j+1} (\mathrm{mod}\ 2), \qquad i > 1. \tag{1}$$

For $i = 1$ we will substitute

$$a_{1,j} = a_j, \qquad 1 \leqslant j \leqslant n. \tag{2}$$

There are 2^n possible different first rows, which may be interpreted as all binary numbers $<2^n$. These 2^n triangles together contain

$$\frac{n(n+1)}{4} 2^n \text{ digits } 1,$$

so that on the average there is given the number asked for.

The number of digits 1 of a triangle will be denoted by

$$A(n) = A(a_1, \ldots, a_n).$$

Since it follows with (1)

$$a_{i-1,j} + a_{i,j} \equiv a_{i-1,j+1} \text{ and } a_{i-1,j+1} + a_{i,j} \equiv a_{i-1,j} \ (\mathrm{mod}\ 2), \tag{3}$$

every triangle may be read from three sides, which themselves may be read from the right or the left, so that in general there are six possibilities. This means

$$\begin{aligned} A(a_1, \ldots, a_n) &= A(a_n, \ldots, a_1) = A(a_{1,1}, \ldots, a_{n,1}) \\ &= A(a_{n,1}, \ldots, a_{1,1}) = A(a_{1,n}, \ldots, a_{n,1}) = A(a_{n,1}, \ldots, a_{1,n}). \end{aligned} \tag{4}$$

In the following every congruence is to be considered modulo 2, if no other notation is used. For $a_{i,j}$ we get

$$a_{i,j} \equiv \sum_{\nu=0}^{i-1} \binom{i-1}{\nu} a_{j+\nu}, \tag{5}$$

and this leads to

$$A(n) \equiv \sum_{i=1}^{n} \sum_{j=1}^{n-i+1} \sum_{\nu=0}^{i-1} \binom{i-1}{\nu} a_{j+\nu}. \tag{6}$$

To reduce the congruences in (6) for lower values of n it is perhaps easier to use $A(1) = a_1$ and the recurrence formula

$$A(n) = A(a_1, ..., a_n) \equiv A(a_1 + a_2, a_2 + a_3, ..., a_{n-1} + a_n) + \sum_{j=1}^{n} a_j . \quad (7)$$

There is exactly one triangle with $A(n) = 0$: $a_j = 0$, $1 \leqslant j \leqslant n$. The next possible number is $A(n) = n$: $a_j = 1$, $1 \leqslant j \leqslant n$. Because of (4) there are two further triangles with $A(n) = n$. The greatest number of digits 1 being possible is

$$A(n) = \left[\frac{n(n+1)+1}{3}\right]:$$

$a_j = 0$ for $j \equiv 0 \pmod 3$ and $a_j = 1$ for $j \equiv 1, 2 \pmod 3$. On account of (4) there are three different triangles having this number if $n \not\equiv 1 \pmod 3$ and two otherwise. In order to prove the upper bound for $A(n)$ we consider all possible triangles with $n = 2$:

$$\begin{matrix} 0\ 0 & & 0\ 1 & & 1\ 0 & & 1\ 1 \\ 0 & & 1 & & 1 & & 0 \end{matrix}$$

We see that every triangle has at most two digits 1. Now every partial triangle with $m = 2$ of the above constructed one has exactly two digits 1, so that the total number cannot be greater. So we can write

$$0 \leqslant A(n) \leqslant \left[\frac{n(n+1)+1}{3}\right] = \left[\frac{2N+1}{3}\right], \quad (8)$$

where $[x]$ means the greatest integer not exceeding x.

In the following we will construct triangles with $A(n) = \frac{1}{2}N$ for all $n \equiv 0, 3 \pmod 4$.

At first we seek for a class of triangles with $n = pk$, $k = 1, 2, ...,$ being of the kind that the first rows consist of k p-tuples of digits, that is,

$$a_j = a_{j+\nu p}, \qquad 1 \leqslant j \leqslant p, \qquad 1 \leqslant \nu \leqslant k-1, \quad (9)$$

and the p congruences

$$a_j = a_{p+1,j} \equiv \sum_{\nu=0}^{p} \binom{p}{\nu} a_{j+\nu}, \qquad 1 \leqslant j \leqslant p, \quad (10)$$

are conditions for $a_1, a_2, ..., a_p$ (see Figure 2 without the broken lines). If

a denotes the number of digits 1 for the vertical shaded parts and b the number for the remaining not broken lined parts in Figure 2, then

$$A(n) = \frac{a+b}{144} N + \frac{11a - 13b}{288} n, \qquad n = 12k, \qquad k = 1, 2, \ldots . \quad (11)$$

For $p = 1, 2, 4, 5, 8, 9, 10, 11$ the congruences (10) only have the trivial solution $a_j = 0$, $1 \leqslant j \leqslant p$. So we choose $p = 12$, the first value being divisible by 4 and leading to the non-trivial conditions

$$a_j + a_{j+4} + a_{j+8} \equiv 0, \qquad 1 \leqslant j \leqslant 4. \quad (12)$$

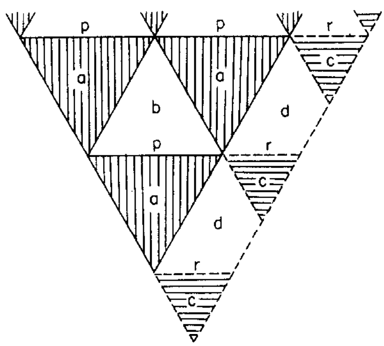

FIGURE 2

Four triples 000, 011, 101, and 110 fulfill every congruence (12), so that there are $4^4 = 256$ possible classes of triangles asked for. None of them has the number $A(n) = N/2$ for all k. This is to be seen already for $k = 1$: By (7) follows

$$A(12) \equiv a_2 + a_3 + a_6 + a_7 + a_{10} + a_{11}, \quad (13)$$

and together with (12) this leads to $A(12) \equiv 0$ contradicting $(12 \cdot 13)/4 = 39$. But in any case for 60 triangles $(a + b)/144 = 1/2$ holds, that means, $A(n)$ differs from $N/2$ only in a summand, being linear with n. Now we will try to correct this term for at least one of the above 60 cases.

We consider $n = 12k + r$, $r = 3, 4, 7, 8, 11, 12$, and the broken lined part of Figure 2. If we are successful in finding r digits a_{n-j} with (use (10))

$$a_{n-j} = a_{13,n-j-12} \equiv \sum_{\nu=0}^{3} a_{n-j-12+4\nu}, \qquad 0 \leqslant j \leqslant r - 1, \quad (14)$$

so that every horizontal shaded part of Figure 2 contains $c = r(r + 1)/4$ digits 1, and every remaining trapezoid has $d = (r/4)(23 - r) + 39 - a$

digits 1, then $A(n) = N/2$ for all $n = 12k + r$. So we have to check up $60B(r)$-times the validity of $a + 2c + d = N/2$ for $k = 1$, $B(n)$ being the number of different triangles with $A(n) = N/2$. The number of tests will be reduced by (14) and (7). In accordance with (9) we denote

$$a_{n-j} = a_{r+12-j}\,, \qquad 0 \leqslant j \leqslant r - 1. \tag{15}$$

Then (14) become equal (12) for $r - 4 \leqslant j \leqslant r - 1$. Using (12) the rest of (14) gives, however,

$$a_{13+j} = a_{1+j}\,, \qquad 0 \leqslant j \leqslant r - 5, \qquad r \geqslant 5, \tag{16}$$

being restrictive conditions. The congruence $A(n) \equiv N/2$ for $k = 1$ rekursive determined by (7) together with (12), (16), and

$$A(a_{13}\,, a_{14}\,,..., a_{12+r}) \equiv \frac{r(r+1)}{4}$$

gives the further condition

$$a_{r+9} \equiv a_{r-3} + 1, \qquad r = 4, 7, 8, 11, 12, 15. \tag{17}$$

We now start with $r = 4$.

$n = 12k + 4$: $B(4) = 6$. We find the following 38 different first rows. The 12-tuple of digits has to be repeated k-times.

000001110111...1010	101000101000...0100
000001110111...1100	101010110001...0101
000101000101...1010	101100011010...0100
000101000101...1100	101101001111...0011
001000010011...1100	101101001111...0100
001010001010...1100	101110000011...0011
001100100001...1010	101111110100...0011
001111001111...1010	101111110100...0100
010001010001...1010	110000011101...0100
010010111111...1010	110010000100...0101
010010111111...1100	110011110011...0101
010100010100...1100	110100101111...0010
010110001101...1010	110101011000...0010
010111111010...1100	110101011000...0101
011000110101...1100	110111000001...0101
100001001100...0011	111100111100...0100
100001001100...0101	111101001011...0011
100010100010...0101	111110100101...0010
100011010101...0010	111110100101...0100

$n = 12k + 3: B(3) = 4$. The congruences $A(15) \equiv a_{13} + a_{14} + a_{15} \equiv 0$ and $A(a_{13}, a_{14}, a_{15}) \equiv a_{13} + a_{14} + a_{15} \equiv 1$ give a contradiction so that there is no solution of this kind for $r = 3$. If we, however, test the triangles found above for $r = 4$, whether the first row has $6k + 2$ digits 1, then we are successful 8-times and get solutions by rejecting these rows. The 12-tuple has to be repeated $(k - 1)$-times.

000010011001...000010011000010	101001011111...101001011110010
000010011001...000010011000111	110010000100...110010000101010
011001000010...011001000011111	111010010111...111010010110111
011111101001...011111101000111	111111010010...111111010011111

It is easy to see, that these solutions are of the type shown in Figure 2 with $r = 15$. The triangle determined by the last three digits is overlapping, so that $A(a_{25}, a_{26}, a_{27}) = 3$ must hold. This leads to further 26 solutions. In all we so have got 34 triangles with $A(n) = N/2$ for $n \equiv 3 \pmod{12}$, $n \geqslant 15$.

000001110111...000001110110001	011100000111...011100000110010
000011101110...000011101111010	011101110000...011101110001010
000011101110...000011101111100	011111101001...011111101000100
000101000101...000101000100100	100000111011...100000111010100
000101000101...000101000100111	100101111110...100101111111100
000110101011...000110101010001	101010110001...101010110000100
000111011100...000111011101001	101011000110...101011000111100
001011111101...001011111100001	101100011010...101100011011100
001110111000...001110111001001	101110000011...101110000010001
011000110101...011000110100001	110000011101...110000011100111
011000110101...011000110100100	110001101010...110001101011111
011001000010...011001000011001	110100101111...110100101110001
011010101100...011010101101010	110111000001...110111000000111

$n = 12k + 7: B(7) = 12$. We obtain 7 different first rows.

010000100110...0101011	101011000110...1011111
010101100011...0100001	110001101010...1101010
011010101100...0111100	111011100000...1111101
100100001001...1000010	

$n = 12k + 8: B(8) = 40$. There are 20 solutions. But none of them gives a solution for $r = 7$ by rejecting the first row.

000001110111...00001011	100001001100...10001010
000001110111...00001101	101000101000...10101111
010001010001...01001000	101100011010...10111011
010010111111...01000011	101110000011...10110000
010011001000...01000010	101111110100...10110000
010011001000...01000101	110011110011...11000100
010100010100...01011010	110100101111...11011101
010110001101...01010111	110101011000...11011101
010111111010...01010001	110111000001...11010000
010111111010...01010110	111110100101...11110011

$n = 12k + 11$: $B(11) = 171$. The following 18 triangles have been found.

001001100100...00100111000	011010101100...01101011111
001011111101...00101110001	100101111110...10010110000
001101010110...00110100001	100101111110...10010110011
001110111000...00111010000	100110010000...10011000010
010000100110...01000011001	101011000110...10101101010
010001010001...01000100100	110001101010...11000111100
010100010100...01010000101	111000001110...11100001101
010100010100...01010000110	111010010111...11101000100
010101100011...01010111011	111010010111...11101000111

$n = 12k + 12$: $B(12) = 410$. We get 18 first rows. One of them being rejected gives a solution for $r = 11$.

000101000101...000101001000	010111111010...010111110111
000101000101...000101001110	011000110101...011000111111
010001010001...010001011010	011001000010...011001001110
010001010001...010001011100	101000101000...101000100101
010010111111...010010110011	101001011111...101001010100
010011001000...010011000011	101100011010...101100010000
010100010100...010100010100	110010000100...110010001001
010100010100...010100011111	110011110011...110011111000
010110001101...010110000111	110111000001...110111000001

Summarizing for every $n \equiv 0, 3 \pmod 4$ there exist at least 4 triangles containing as many digits 1(−) as 0(+).

Reference

1. H. Steinhaus, "One Hundred Problems in Elementary Mathematics," Pergamon, Elinsford, N.Y., 1963, pp. 47–48.